XPERIENCE OF DISABLED-WORKER BENEFITS UNDER OASDI, 1965-74

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Social Security Administration . . . . . Office of the Actuary

ACTUARIAL STUDY NO. 74

JANUARY 1977

DHEW Pub. No. (SSA) 77-11521

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A. Introduction

This Actuarial Study presents data on the incidence and termination experience for disabled-worker benefits under the Old-Age, Survivors, and Disability Insurance system (OASDI). Various Actuarial Notes have previously presented data pertaining to the incidence of disability among insured persons, and a previous Actuarial Study has presented data regarding terminations.[1/] The most recent of these publications, however, was in 1969.

The primary purpose of this study is to present recent information on incidence and termination rates for the disability insurance program. These rates are an important basis for projection of the cost of the disability insurance program. In order to project the future cost of the program, some hypotheses as to the future course of the incidence and termination rates must be made. This in turn involves the adoption of some hypotheses as to the causes of past trends. In the absence of proof to the contrary we have adopted some hypotheses that can be considered plausible. It should be emphasized, however, that rigorous proof of the hypotheses is lacking (and, in fact, may be impossible) and that several knowledgeable persons within the Social Security Administration are in disagreement with us on some of the hypotheses, or with the emphasis placed on some of the hypotheses.

Included as an appendix to this study is a derivation of a two-dimensional Whittaker-Henderson Type B graduation formula by Steven F. McKay and John C. Wilkin. It is believed that this is the first time such a formula has been used. It greatly facilitated the graduation of the select tables of disability termination rates.

1/ Actuarial Notes No. 18, 36, 45, and 58 provide analysis of earlier disability incidence experience, and Actuarial Study No. 65 provides an analysis of earlier disability termination experience.

B. Analysis of Data on Incidence of Disability

An important cost factor in the disability insurance program is the rate at which insured workers become disabled and qualify for monthly disability benefits. This rate is generally referred to by actuaries and demographers as the "disability incidence rate".

The rate of disability allowance (the proportion of the claims filed that is awarded benefits) is not significant from a cost standpoint. This rate depends upon how many non-qualifying claims are filed, which may be affected by extraneous factors. As an example, persons with non-disabling physical impairments would be likely to file claims at the time they lose their jobs in a recession. Since disability is defined as the "inability to engage in any substantial gainful activity" it is doubtful how many of these persons should be allowed benefits. One must question the alleged disability of a person who has had no change in his condition since he last actually engaged in substantial gainful activity. As a second example, it is often suggested that greater public awareness of the DI program will increase claims. A large segment of the population has been unaware of the Disability Insurance program and it is possible that some newly impaired persons did not file a claim promptly because of this unawareness. We believe, however, that very few totally disabled persons have failed to file a claim because of unawareness of the availability of benefits. It is hard to conceive of a totally disabled person who for a long period of time failed to seek the assistance of a relative, a friend, a doctor, or a state or local agency, none of whom informed him of the disability program. The argument of increased public awareness of the Disability Insurance program rests not only on the worker's ignorance, but also on his total lack of communication with others, something that is very unlikely of a relatively young person (under 64) who somehow has been managing to survive without any income.

Table 1 presents a brief analysis of the approximate gross disability incidence rates that have been observed from 1957-75. The approximate gross disability incidence rate as used in Table 1 is defined as the total number of disabled-worker benefits awarded in a calendar year (all ages and both sexes combined) divided by the number of workers insured for disability at the beginning of the year (including those already receiving disability benefits). This is an approximation to the more precise definition of disability incidence rate as the number of entitlements to disability benefits in a year divided by the average number of insured workers in the year (excluding those already entitled to disability benefits). This is the only table in which data are recorded according to year awarded; all other tables are by year of entitlement. This table, although simple and crude, gives a good indication of the past trend in disability incidence rates.

The term "entitlement", as used by the OASDI program, has a very special meaning. Entitlement occurs when all of the conditions of eligibility are satisfied (such as age, insured status, impairment, and waiting period), provided that a timely claim has been filed. Entitlement does not depend on when the determination of disability or "award" is made. When the actual award is made after the month of entitlement, benefits are paid retroactively to the month of entitlement.

The gross disability incidence rate was initially high, due mainly to the fact that many of the benefits awarded in the early years of the program were to persons who were disabled before the inception of the program, rather than to persons who had recently become disabled. Another important factor was that only workers over age 50 could qualify for monthly benefits then, and disability incidence rates are much higher at those ages.

The gross disability incidence rate dropped significantly in 1960 with the elimination of the age 50 limitation. However, the full effect of this change in the program was not felt for several years, because of the time lag in processing the awards of the newly eligible group. In fact more awards were made to this group in 1961 than in 1960, that is why the 1960 rate was lower than the 1961 rate.

In 1965, after the effects of the elimination of the age 50 limitations had become stable, the definition of disability was liberalized. The change in definition (from permanent disability to disability of at least 12 months expected duration) had the effect of increasing the gross disability incidence rates. As before, the full effect of this change was not felt immediately, because of the time lag involved in processing the cases of newly eligible persons.

A provision in the 1967 Social Security Amendments liberalized the insured status requirements for persons under age 31. 2/ This made many more young persons eligible for disability benefits, thus lowering the gross incidence rate (by bringing into the insured group a large number of young workers with lower incidence rates).

2/ Insured status is defined by means of quarters of coverage (calendar quarters in which a worker has been paid at least $50 in earnings from covered employment). To be fully insured, a worker must have one quarter of coverage for each calendar year elapsing after attainment of age 21 (or the year 1950 if later) and prior to the onset of disability. To be disability insured, a worker must be both fully insured and recently connected to

The rapid and continuous increase in the gross incidence rate since 1970 cannot be explained in terms of legislated changes in the disability program, except for the small increase that may be attributable to the change in the waiting period from 6 months to 5 months in the 1972 Amendments. 3/

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the labor force. Prior to the 1967 amendments, all workers needed 20 quarters of coverage out of the latest 40 calendar quarters ending with the calendar quarter of disability, to be recently connected. Since the enactment of the 1967 amendments, those workers under age 31 at onset of disability have been subject to a less restrictive definition of recent connection to the labor force. That is, they must be covered in no less than half the calendar quarters elapsing after the calendar quarter of attainment of age 21, and in no case less than 6 of the last 12 quarters, to be recently connected.

3/ The law as amended in 1972 makes provision for a 5-month waiting period. However, in practice this period is actually 6 to 7 months because it is measured from the beginning of the calendar month following the month of onset, and the beneficiary must live until the end of the calendar month following the 5-month waiting period to be entitled to his first benefit check.

We believe that part of the recent increase in incidence rates is due to the rapid rise in benefit levels since 1970, particularly when measured in terms of pre-disability earnings. From December 1969 to December 1975 there were general benefit increases amounting to 82%. Also, effective in 1973, Medicare benefits became available to disabled worker beneficiaries who have been entitled for at least two years. We also believe the short computation period for the young workers, the weighting of the benefit formula for the low income workers, and the additional benefits payable when the worker has dependents can provide especially attractive benefits to beneficiaries in these categories. It is possible under the present formula for these beneficiaries to receive more in disability benefits than was included in their take-home pay while they were working. 4/ Benefits this high become an incentive to file a claim for disability benefits, and to pursue the claim through the appellate procedures.

Another factor in the recent increase in incidence rates is the high unemployment that the country has experienced since 1970. Physically impaired individuals are more likely to apply for disability benefits if they lose their jobs in a recession than during an economic expansion when they can retain their jobs. A certain portion of these additional applications will meet the requirements for benefits. Although it is doubtful how much the economy should influence the incidence rates, as was discussed earlier, to the extent that it does, one must keep in mind that the rates experienced by the OASDI program prior to 1970 took place in a relatively strong economic environment while the rates experienced after 1969 took place in a weakening economic environment.

Recent increases in incidence rates may also have been influenced by changes in attitude in the population. It is possible that the impaired lives of today do not feel the same social pressure to remain productive as did their counterparts as recently as the late 1960's. This could radically change their perception of themselves as needing the help of others through a government program. This social pressure is influenced not only by how society views the impaired, but also by how society views people who rely on social programs for their livelihood.

All of these factors depend, in part, on the problem of the determination of disability. John Miller, a consulting actuary and recognized expert in the field of disability insurance, described this problem as follows:

4/ For a table presenting selected replacement ratios, see Table 2 of "Disability Experience Under the Social Security Program", a paper presented for discussion at the annual meeting of the Society of Actuaries, October 20-22, 1975 by A. Haeworth Robertson, Chief Actuary of the Social Security Administration. This paper is published in the RECORD of the Society of Actuaries, Volume I number 4.

"The underlying problem in providing and administering any plan of disability insurance is the extreme subjectivity of the state of disability. This characteristic could be discussed at length and illustrated with an almost endless array of statistics but it can best be visualized by comparing a Helen Keller or a Robert Louis Stevenson with any typical example of the multitude of ambulatory persons now drawing disability benefits who could be gainfully employed if (a) the necessary motivation existed, and (b) an employment opportunity within their present or potential capability were present or made available. Thus the problem is not simply one of medical diagnosis. The will to work, the economic climate and the "rehabilitation environment" outweigh the medical condition or problem in many, if not in most, cases." 5/

It is believed that the above factors are responsible for a large part of the increases in the gross incidence rate, however, it is doubtful that they can fully account for the rather rapid increase that has been observed. We feel that some administrative factors must have also played an important part in the recent increases, but we cannot offer a definite proof to that effect.

Although many disability programs, both public and private, have had trouble holding down program costs in recent years, many administrative factors of the social security disability program make strict control of the program particularly difficult to achieve. Among these factors are the multi-step appeals process, enabling the claimant to pursue his case to the "weak link" in the hierarchy of disability determination; the massive nature of the system, necessitating action on over one million claims per year; and in general, the difficulty of maintaining a proper balance between sympathy for the claimant and respect for the trust funds in a large public system.

The multi-step appeals process provides that a claimant who has been denied benefits may request first a reconsideration, then a hearing before an Administrative Law Judge, appeal his hearing to an Appeals Council, and ultimately take his claim to court. By

5/ Reports of Consultants on Actuarial and Definitional Aspects of Social Security Disability Insurance, to the Subcommittee on Social Security of the Committee on Ways and Means, U.S. House of Representatives, P. 24.

the very nature of the claims process, the cases which progress through the appeals process are likely to be borderline cases where vocational factors play an important role in the determination of disability. As stated previously, the definition of disability as defined by the Social Security Act is "inability to engage in any substantial gainful activity by reason of a medically determinable impairment...". William Roemmich, M.D., former Chief Medical Director of the Bureau of Disability Insurance has observed that there are two significant variables in this definition of "inability to engage:" (1) impairment and (2) vocational factors. And that a heavy emphasis on vocational factors can change the definition to "inability to engage in usual work by reason of age, education, and work experience providing any impairment is present." 6/ Therefore, to the extent that vocational factors are given higher weight as a claim progresses through the appeals process, the chances of reversal of a former denial is increased.

The increasing importance of vocational factors in making disability determinations is to a certain extent beyond the control of the Social Security Administration. The Kerner vs Flemming decision in 1960 stipulated that once an individual has established that he cannot perform his usual work the burden is on the Government to produce evidence to show what jobs an individual can perform. The 8th circuit court went one step further. In Garrett vs Richardson (1972) the court required hearing examiners to produce a vocational counselor to testify on the issue of substantial gainful activity.

As an indication of the growing importance of the appeals process in recent years, from fiscal year 1969 to fiscal year 1973, the number of DI hearings processed increased from 28,000 to 56,000 and the reversal rate increased from 39% to 46%.

Not only is the massive nature of the disability determination process difficult to conceptualize, but so also is the phenomenal growth in the burden that has been placed on the system in recent years. In fiscal year 1969, the Social Security Administration took in over 700,000 claims for disability insurance benefits. By fiscal year 1974, not only was the SSA taking in over 1,200,000 disability insurance claims per year, but it had also taken in over 500,000 disability claims under the black lung program which started during fiscal year 1970, and it was taking in over 1,000,000 disability claims per year under the Supplemental Security Income program which started in fiscal year 1974. All

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6/ Ibid., p. 65.

of this happened during a period when the administration was making a concerted effort to hold down the administrative costs of the program.

All of this put tremendous pressure on the disability adjudicators to move claims quickly. As a result the administration reduced their review procedures to a small sample, limited the continuing disability investigations on cases which were judged less likely to be terminated, and adopted certain expedients in the development and documentation in the claims process. Although all of these moves may have been necessary in order to avoid an unduly large backlog of disability claims, it is our opinion that they had an unfortunate effect on the cost of the program.

By claiming that it is difficult to maintain a proper balance between sympathy for the claimant and respect for the trust funds, we do not mean that disability adjudicators consciously circumvent the law in order to benefit an unfortunate claimant. What is meant is that in a public program designed specifically to help the people, such as Social Security, whose operations are an open concern to millions of individuals, and where any one decision has an insignificant effect on the overall cost of the program, there is a natural tendency to find in favor of the claimant in close decisions. This tendency is likely to result in a small amount of growth in disability incidence rates each year, such as that experienced under the DI program prior to 1970, but it can become highly significant during long periods of difficult national economic conditions.

Presented in Tables 2 and 3 are the results of a study of disability incidence rates. These rates are the number of entitlements in each year 1965 through 1973, based on the benefit awards recorded before October 1975, divided by the average number of insured workers during the year (excluding those already entitled to disability benefits).

The incidence rates resulting from calculations based on year of entitlement are slightly different from those based on year of onset. These represent two different concepts of incidence of disability benefits. The former refers to the year for which benefit payments start, while the latter refers to the year in which the individual becomes sufficiently impaired to qualify for benefits (regardless of the waiting period or of the time elapsed before the first benefit payment was made). Most analysts prefer to work with values based on the "onset concept", mainly because to them the important event is the occurrence of the impairment. Some analysts prefer to work with values based on the "entitlement concept", mainly because to them the important event is the

payment of benefits. We believe that the "onset concept" allows for a slightly better level of homogeneity in the data needed for studies on termination of benefits, but that in practical applications the "entitlement concept" may permit a substantially higher level of accuracy in the data. The rates calculated under the "entitlement concept" also have greater applicability in making cost estimates. In the net we prefer the "entitlement concept."

One very striking effect of the use of year of entitlement basis occurred as a result of a provision in the 1972 Amendments which changed the waiting period from 6 months to 5 months. As a result, entitlements in January 1973 reflect onsets of disability that occurred both 6 months and 5 months earlier. This explains a large part of the increase in incidence rates from 1972 to 1973 as shown in Table 3.

Table 2 shows the number of disability benefit awards made before October 1975 by sex and age for each single year of entitlement from 1965 to 1973. These data, obtained on a stratified sample basis, represents over 3 million awards. The sample ranges from 10% to 100%, depending on how the award is classified.

The disability incidence rates in this analysis are slightly incomplete in that they are based only on disability benefit awards actually made before a certain date (October 1975) and do not include awards made after that date with entitlement prior to 1974. Obviously, for any particular year of entitlement, there will be some additional cases which are awarded after October 1975. This happens because of delays in filing claims, delays in processing claims and because of reconsideration of original denials and because of all the delays involved in the appellate process. It is believed that this limitation results in disability incidence rates that are understated by perhaps 5% in 1973, by about 1% in 1972, and by insignificant amounts for earlier years.

Table 3 shows the disability incidence rates by age and sex for years of entitlement 1965 to 1973 separately.

Table 4 shows an estimate of the experience for 1975. This estimate is based on the actual detailed data through 1973 (taking into account the effect of the change to a 5-month waiting period in 1973) and on the gross award data (not classified by age, sex, or year of entitlement) through the end of 1975. As would be anticipated, the rates rise steadily and significantly with advancing age. The increase in incidence rates from the age group 55-59 to the age group 60-64 would be much more pronounced if it were not for the availability of actuarially reduced old-age benefits beginning at age 62. The availability of these benefits results in

many beneficiaries electing old-age benefits rather than attempting to obtain disability benefits, since the latter requires a waiting period, involves a slower and more complicated adjudication process and includes the possibility of denial.

The age-specific disability incidence rates for women are significantly lower than those for men. At the youngest ages, the female rates are only about 40% of the male rates; at the middle ages, they are about 90% of the male rates; while at the age group 60-64 they are about 60% of the male rates. When comparing these incidence rates between men and women it must be kept in mind that insured men and women are far from homogeneous as to occupational risks; there being few women engaged in the hazardous businesses where the incidence of disability is higher.

C. Analysis of Data on Terminations

There are two types of disabled-worker benefit terminations of actuarial importance: death of the beneficiary and recovery of the beneficiary from the disability. The experience for both are discussed in this Actuarial Study. Other types of terminations either are peculiar to OASDI only, such as age 65 termination,7/ or are of very limited over-all significance.

Table 5 presents a brief analysis of the gross termination rates that have been observed from 1957 to 1975. The gross termination rate is defined as the ratio of the number of terminations in a year to the average number of benefits in force in the year. The gross death termination rate has been decreasing almost continuously since the beginning of the program. Part of this decrease is due to changes in the program and to the maturation of the program. The rate was high initially, since the program then was limited to workers age 50-64. The elimination of the age-50 limitation in the 1960 Amendments brought in many young disabled workers with correspondingly lower mortality.

The liberalized definition of disability in the 1965 Amendments from permanent disability to one that is expected to last at least 12 months brought in many disabled workers who were expected to recover from their disability rather than remain disabled until their death. It is believed that these beneficiaries would experience lower death rates.

The liberalization of the insured status requirement for persons under age 31 in the 1967 Amendments brought in more younger disabled workers, who normally experience lower mortality.

The maturation of the program also contributed to the decline in the gross death termination rate. As the program grows older there is a reduction in the concentration of disabled-worker beneficiaries at the shorter duration periods, where mortality is generally higher.

The same factors that have contributed to the decline in the death rates in the general population, particularly improved medical procedures, could also account for some of the decline in the death rates of the disabled. The gross death rate for the total population has gone from 9.5 per thousand in 1960 to 8.9 per thousand in 1975, a decrease of 6 percent. Over the same period, however, the gross death rate for disabled-workers has gone from 109.6 per thousand to 58.5 per thousand, a decrease of 47 percent.

7/ According to administrative procedures followed under OASDI, benefits to disabled workers are terminated upon attainment of age 65, and old-age retirement benefits are automatically awarded to replace them.

Although all of these reasons have contributed to the decline in the gross death termination rate, it is doubtful that they can fully account for the rather rapid decrease that has been observed. It is believed that healthier applicants are being awarded disability benefits and consequently there is a tendency for the overall mortality rates to decline. This belief is based, at least in part, on the fact that age-sex specific disability rates have nearly doubled since 1965, and we do not believe that this has happened because of changes in the average status of health in the nation. The magnitude of the increase in the incidence rates is so substantial, that it is likely to have had a significant effect on the characteristics of applicants that are being awarded disability benefits. It is our belief that progressively healthier individuals have been granted benefits, and that progressively healthier individuals have been allowed to stay on the rolls.

Initially, the gross recovery rate was very low, since immediate recovery of an individual who has just been found to be permanently disabled is very unlikely. During the first few years of the program, the rate grew as beneficiaries had more time to recover.

The gross recovery rate has been significantly affected by legislative changes in the DI program. In the 1960 Amendments, a trial-work-period was initiated. 8/ The immediate effect was to delay some recoveries, but by 1962 when the full effect of the trial-work-period on the number of recoveries was felt, the recovery rate began increasing. In addition, the elimination of the age-50 limitation for benefit payments brought in younger workers with higher recovery rates. This also tended to increase the overall recovery rate for all disabled workers.

The effect of some of the provisions in the 1965 Amendments was to increase the number of recoveries. Certain provisions allowed trust fund monies to be spent on the rehabilitation of disabled worker beneficiaries. The full effect of these provisions was not felt until 1967, when the recovery rate reached its highest level. Also, the liberalized definition of disability brought in many healthier disabled workers who were expected to recover some time after the first 12 months of disability.

The rapid decrease in the gross recovery rate since 1967 cannot be explained in terms of legislated changes since there have not been any major changes in the law since then. As with the decline in the gross death rate, and probably even more so, it is believed that progressively healthier beneficiaries are being allowed to continue receiving benefits without being terminated.

8/ This provision encourages beneficiaries to work by permitting trial work for 9 months without loss of benefits. Thereafter, benefits may be terminated because of recovery, as demonstrated by substantial gainful activity, even though the medical impairment is unchanged.

In the administration of the OASDI disability program, benefits are terminated due to recovery if it is determined that the beneficiary can engage in substantial gainful activity. There are two types of determinations which can lead to this conclusion. The first is a determination that, although the beneficiary is not engaging in substantial gainful activity, the physical or mental condition of the beneficiary has improved to such an extent that he is capable of engaging in substantial gainful activity. The second is a determination that the beneficiary has demonstrated his ability to engage in substantial gainful activity by a return to work, regardless of his medical condition.

It is believed that there is a possibility the number of recoveries due to a determination of improvements in the beneficiary's physical condition has declined since 1967 because of administrative expediency. As was mentioned earlier, the high workload pressures of recent years has forced the administration to curtail some of its policing activities. Prior to 1970 the Social Security Administration made about 140,000 continuing disability investigations per year. This meant that about 10% of the DI beneficiaries were being investigated in any year. During fiscal years 1971 to 1974, when the administrative crunch of the black lung and SSI programs were at their peak an average of a little over 80,000 continuing disability investigations were made per year. This meant that there was an investigation on just over 4% of the DI beneficiaries in a year.

It is believed that the number of recoveries due to the disabled workers returning to work is greatly influenced by the level of benefits; particularly as this is measured by replacement ratios. One possible definition of a replacement ratio that is useful for illustrative purposes pertains to steady workers whose earnings increase at the same rate as the median. In this case, the replacement ratio is computed as the annual amount of benefits received by the disabled worker and his dependents divided by his after tax earnings in the year before onset of disability. On this basis the average replacement ratio of disabled workers with median earnings has increased from about 60% in 1967 to over 90% in 1975, an increase of about 50%. During this time the gross recovery rate has decreased to only 1/3 of what it was in 1967.

The gross termination rate due to both death and recovery combined has been decreasing from initial values of 15%-16% per year and are currently at values of about 7%-8%--or about a 50% decrease during the period.

Historically, benefit termination rates for disabled persons have been found to be heavily dependent on the duration of the disability as well as on the age of the person. This has again

been found to be the case in the experience of disabled-worker beneficiaries under OASDI, as can been seen from Tables 8 to 11.

The termination rates by age, sex, and duration presented here are based on the termination experience that was recorded before October 1975 for anniversaries in the period 1968-74. Anniversaries are measured from the month of entitlement to disability benefits, rather than the usual month of onset of disability. Before 1973, when a 6-month waiting period was applicable, entitlement normally occurred about six and one-half months after onset of disability. A 5-month waiting period was made effective in January 1973, due to a provision in the 1972 Amendments. Therefore, entitlements after 1972 normally occurred about five and one-half months after onset of disability.

The most common situations where entitlement and onset of disability are separated by periods of different lengths than the waiting period are (1) where a potential beneficiary significantly delays his claim for benefits, or (2) where there has been a previous entitlement to disability benefits by the beneficiary. In the second situation, no waiting period is required and entitlement normally coincides with the new onset. In the first situation, OASDI benefits may be paid in event of late filing, to cover elapsed periods of disability up to 12 months, and therefore entitlement may occur at almost any month after the end of the waiting period, depending on the length of time between onset of disability and filing. It is believed that neither of these situations occurs often enough to significantly distort the calculated rates.

The small theoretical loss of homogeneity in measuring durations from entitlement as compared to measuring from onset of disability is more than offset in actual practice by the greater applicability of the final rates in making cost estimates, and the much greater accuracy with which the needed information is recorded, as was mentioned in the previous section.

A total of 724,410 terminations is included in this study. This is distributed as follows: 453,641 male deaths, 115,439 female deaths, 126,624 male recoveries, and 27,706 female recoveries. It was decided to prepare graduated termination rate tables for deaths and recoveries separately, but not for the total. 9/

9/ The total termination rates $q^{(t)}_{[x]+n}$ can be obtained from the death rates $q^{(d)}_{[x]+n}$ and the recovery rates $q^{(r)}_{[x]+n}$ by using the formula,

$$q^{(t)}_{[x]+n} = 1 - [\ (1-q^{(d)}_{[x]+n})\ \ (1-q^{(r)}_{[x]+n})\ ].$$

Observed termination rates were calculated by single years of age for ages 20 to 64, and for each integral year of duration 0 to 4 and for durations 5 and over. The graduations were performed by a two-dimensional Whittaker-Henderson Type B formula.

The horizontal and vertical coefficients of smoothness were chosen, after several trials, so as to obtain a smooth pattern in the rates while deviating as little as possible from the actual. The resulting ratios of actual terminations to expected terminations are shown in Tables 6 and 7.

It should be noted that since duration is measured from the month of entitlement, the rates for all durations, including duration zero, are based on the usual definition of one year probabilities.

There is a curious decrease in mortality around age 55 for duration zero. It is believed that this could be due to a tendency for progressively including more vocational factors (as opposed to physical and medical factors) in determining disability benefit awards to older workers, which would mean that older disabled workers are relatively healthier. These rates increase rapidly again at age 61, which is probably due to the availability of old-age benefits beginning at age 62. The availability of these benefits would result in healthier lives opting for the certainty and promptness of old-age benefits, rather than applying for disability benefits with the possibility that the benefit might be denied and which would be delayed due to the waiting period and to the longer time that it takes to process a claim for disability benefits.

The graduated death and recovery terminaton rates for females are compared with those for males in Tables 12 and 13. The female mortality rates are higher than the male rates at ages under 30, but in general they are lower than the male rates, as is the case in general population mortality. However, the sex differential in mortality is lower for disabled beneficiaries than for the general population.

The female recovery rates are lower than the male rates at all ages and durations. Also, there appears to be no significant trend by age or duration in the ratio of the female recovery rates to the male rates, with most female rates being about 50%-70% of the corresponding male rates.

In Table 14 the disabled worker mortality is compared with general population mortality according to the United States Life Tables for 1969-71 at similar attained ages. The disabled worker mortality varies from about 3 to 5 times the general population mortality at the higher ages and durations to about 20 to 40 times population mortality at the younger ages and lower durations. In all cases, this ratio is higher for females than for males.

TABLE 1

Number of Awards and Approximate Gross Incidence Rates For Disabled-Worker Beneficiaries, 1957-75

Calendar Year	Number Insured on January 1 (in millions)	Number of Awards During the Year (in thousands)	Approximate Gross Incidence Rate By Year of Award (per thousand)
1957	10.00	179	17.90
1958[1/]	10.36	131	13.79
1959[1/]	11.78	178	13.95
1960	46.36	208	4.49
1961	48.51	280	5.77
1962	50.47	251	4.97
1963	51.52	224	4.35
1964	52.30	208	3.98
1965	53.32	253	4.74
1966	54.99	278	5.06
1967	55.72	301	5.40
1968	67.96	323	4.75
1969	70.12	345	4.92
1970	72.36	350	4.84
1971	74.50	416	5.58
1972	76.14	455	5.98
1973	77.81	492	6.32
1974	80.39	536	6.67
1975	82.01	592	7.22

1/ For statistical purposes the years 1958 and 1959 were defined as covering the periods January 1, 1958 to November 30, 1958 and December 1, 1958 to December 31, 1959, respectively. However, the gross incidence rates are shown after conversion to an annual basis.

TABLE 2

Number of Disability Entitlements by Calendar Year of Entitlement, Age at Entitlement, and Sex, for Awards Made Before October 1975

Calendar Age at Entitlement	Calendar Year of Entitlement								
	1965	1966	1967	1968	1969	1970	1971	1972	1973
					MEN				
Under 25	630	1,186	1,437	11,331	9,591	9,418	9,199	8,933	11,630
25-29	3,452	4,076	4,808	10,501	7,559	8,185	8,763	9,743	12,428
30-34	5,854	6,025	6,696	8,750	7,805	8,922	9,455	10,035	12,664
35-39	8,911	9,071	9,581	10,060	10,563	11,769	12,934	12,710	15,292
40-44	13,468	13,083	14,425	15,794	15,595	17,841	18,541	19,248	22,355
45-49	18,389	19,223	21,167	22,417	23,210	27,279	29,706	30,889	35,947
50-54	28,661	29,753	30,702	33,718	33,517	40,410	45,623	48,320	59,208
55-59	44,332	46,095	47,537	49,983	53,709	62,678	70,789	74,147	89,763
60-64[1/]	36,319	48,589	59,415	66,196	69,447	77,719	87,380	93,053	109,170
Total	160,016	177,101	195,768	228,750	230,996	264,221	292,390	307,078	368,457
					WOMEN				
Under 25	143	135	215	1,983	1,675	1,923	2,199	2,540	3,877
25-29	750	882	1,015	3,085	2,123	2,575	2,970	3,214	4,573
30-34	1,391	1,507	1,699	2,845	2,377	2,526	2,914	3,588	4,828
35-39	2,700	2,670	3,042	3,119	3,231	3,626	4,121	4,104	6,124
40-44	4,338	4,925	5,064	5,483	5,620	5,954	6,895	7,410	9,495
45-49	6,671	7,222	8,502	8,577	9,922	11,543	12,244	13,388	17,558
50-54	11,568	12,499	13,954	14,909	15,882	17,829	20,273	22,480	28,557
55-59	17,731	18,269	19,272	21,055	22,942	26,523	31,474	33,446	43,709
60-64[1/]	12,504	15,381	16,916	19,728	21,255	24,524	27,853	30,770	39,038
Total	57,796	63,490	69,679	80,784	85,027	97,023	110,943	120,940	157,759

1/ Includes the limited number of cases of persons who attain age 65 in the year of entitlement.

TABLE 3

Disability Incidence Rates by Calendar Year of Entitlement, Age at Entitlement, and Sex, Based on Awards Made Before October 1975
(Rates per Thousand)

Calendar Age at Entitlement	Calendar Year of Entitlement								
	1965	1966	1967	1968	1969	1970	1971	1972	1973
					MEN				
Under 25	.3	.6	.7	2.1	1.1	1.0	1.0	1.0	1.2
25-29	.7	.8	.9	1.9	1.2	1.2	1.3	1.3	1.6
30-34	1.2	1.2	1.3	1.7	1.5	1.7	1.7	1.8	2.1
35-39	1.7	1.8	1.9	2.0	2.2	2.4	2.6	2.6	3.0
40-44	2.6	2.5	2.8	3.1	3.1	3.5	3.7	3.9	4.6
45-49	3.8	3.9	4.3	4.5	4.6	5.3	5.9	6.2	7.2
50-54	6.5	6.7	6.8	7.4	7.4	8.8	9.9	10.3	12.6
55-59 1/	11.9	12.3	12.5	13.0	13.7	15.6	17.4	18.2	21.9
60-64 1/	12.7	15.4	18.4	19.9	20.4	22.3	24.8	26.2	30.8
					WOMEN				
Under 25	.2	.2	.3	.7	.3	.4	.4	.4	.6
25-29	.4	.5	.5	1.2	.7	.7	.8	.8	1.0
30-34	.9	.9	1.0	1.7	1.3	1.3	1.4	1.6	1.9
35-39	1.6	1.6	1.8	1.9	1.9	2.1	2.3	2.2	3.1
40-44	2.1	2.4	2.5	2.6	2.7	2.8	3.2	3.4	4.3
45-49	3.1	3.2	3.7	3.6	4.1	4.6	4.8	5.2	6.8
50-54	5.5	5.8	6.3	6.5	6.7	7.3	8.0	8.6	10.7
55-59 1/	9.6	9.6	9.8	10.4	11.0	12.3	14.1	14.6	18.4
60-64 1/	9.5	9.9	10.4	11.5	11.8	13.0	14.2	15.3	19.0

1/ Includes the limited amount of experience before age 65 for persons who attain age 65 in the year of entitlement.

TABLE 4

Estimated Disability Incidence Rates By Age At Entitlement And Sex For Entitlements In Calendar Year 1975

Calendar Age at Entitlement	Rate per Thousand Males	Females
Under 25	1.46	.60
25-29	1.75	1.08
30-34	2.32	1.98
35-39	3.39	3.12
40-44	4.99	4.46
45-49	7.83	6.81
50-54	13.02	11.15
55-59	23.09	18.66
60-64 1/	32.99	19.74

1/ Includes the limited amount of experience before age 65 for persons who attain age 65 in the year of entitlement.

TABLE 5

Number of Terminations and Gross Termination Rates For Disabled-Worker Beneficiaries, 1957-75

Calendar Year	Average Number In Force	Number of Terminations		Gross Termination Rates (per thousand)		
		Death	Recovery	Death	Recovery	Death and Recovery
1957	81,149	8,931	52	110.1	0.6	110.7
1958 1/	201,386	28,099	1,397	152.2	7.6	159.8
1959 1/	288,858	42,771	3,228	136.7	10.3	147.0
1960	397,241 2/	43,543	3,124	109.6	7.9	117.5
1961	539,876	60,538	2,936	112.1	5.4	117.5
1962	684,406	67,020	9,555	97.9	14.0	111.9
1963	789,720	73,344	12,931	92.9	16.4	109.3
1964	866,702	75,812	16,487	87.5	19.0	106.5
1965	948,294	79,823	18,441	84.2	19.4	103.6
1966	1,053,265	84,399	23,111	80.1	21.9	102.0
1967	1,158,987	92,084	37,151	79.5	32.1	111.6
1968	1,258,928	99,924	37,723	79.4	30.0	109.4
1969	1,360,423	108,762	38,108	79.9	28.0	107.9
1970	1,460,007	105,799	40,802	72.5	27.9	100.4
1971	1,586,287	109,883	42,981	69.3	27.1	96.4
1972	1,753,554	108,663	39,393	62.0	22.5	84.5
1973	1,937,430	125,582	36,696	64.8	18.9	83.7
1974	2,141,194	135,083	36,475	63.1	17.0	80.1
1975	2,376,680	138,984	25,366	58.5	10.7	69.2

1/ For statistical purposes the years 1958 and 1959 were defined as covering the periods January 1, 1958 to November 30, 1958 and December 1, 1958 to December 31, 1959, respectively. However, the gross termination rates are shown after conversion to an annual basis.

2/ This figure has been adjusted to take into account the elimination of the age 50 limitation in the year.

TABLE 6

Ratio of Actual Number of Terminations to Expected Number of Terminations[1/] by Sex and Age at Entitlement, 1968-74 OASDI Experience

Calendar Age at Entitlement	Death Terminations		Recovery Terminations	
	Male	Female	Male	Female
Under 25	1.00	1.02	1.00	1.00
25-29	1.00	.97	.99	.98
30-34	.99	1.04	1.01	1.01
35-39	1.00	.97	1.01	.99
40-44	1.00	1.01	1.00	1.01
45-49	1.00	1.01	1.00	1.00
50-54	1.00	.99	1.00	.99
55-59	1.00	1.01	1.00	1.00
60-64	1.00	.99	.92	1.00
Total	1.00	1.00	1.00	1.00

1/ The expected number of terminations is computed according to the actual exposure and the graduated termination rates. The number of terminations were calculated by age and duration and by summing the terminations for durations 4 and under for each age at entitlement with the terminations during the ultimate period assumed to be for entitlement at an age 5 years younger.

TABLE 7

Ratio of Actual Number of Terminations to Expected Number of Terminations by[1/] Sex and Duration, 1968-74 OASDI Experience

Duration	Death Terminations		Recovery Terminations	
	Male	Female	Male	Female
0	1.02	1.01	.98	.96
1	.95	.99	1.03	1.05
2	.99	.96	.96	.94
3	1.02	1.00	.99	.98
4	1.02	1.01	1.01	1.02
5 and Over	1.00	1.01	1.00	1.01
Total	1.00	1.00	1.00	1.00

1/ The expected number of terminations is computed according to the actual exposure and the graduated termination rates. The number of terminations were calculated by age and duration and by summing the terminations for each duration.

TABLE 8

Graduated Select and Ultimate Death Termination Rates
For Male Disabled Workers, 1968-74 OASDI Experience
(per thousand)

x	$q^{(d)}_{[x]}$	$q^{(d)}_{[x]+1}$	$q^{(d)}_{[x]+2}$	$q^{(d)}_{[x]+3}$	$q^{(d)}_{[x]+4}$	$q^{(d)}_{x+5}$	x+5
20	25.7	18.1	12.6	9.5	8.1	7.5	25
21	27.4	19.4	13.7	10.5	9.0	8.3	26
22	29.0	20.7	14.8	11.5	9.8	9.0	27
23	30.7	22.1	15.9	12.4	10.7	9.7	28
24	32.4	23.5	17.0	13.4	11.5	10.4	29
25	34.3	24.8	18.1	14.4	12.3	11.1	30
26	36.2	26.2	19.2	15.3	13.1	11.7	31
27	38.2	27.5	20.2	16.3	13.9	12.4	32
28	40.2	28.9	21.3	17.2	14.8	13.2	33
29	42.1	30.3	22.4	18.2	15.8	14.0	34
30	44.0	31.7	23.6	19.2	16.7	14.9	35
31	45.9	33.3	24.8	20.3	17.8	15.8	36
32	48.0	34.8	26.1	21.5	19.0	17.0	37
33	50.2	36.5	27.5	22.7	20.2	18.3	38
34	52.6	38.3	28.9	23.9	21.5	19.7	39
35	55.1	40.2	30.4	25.3	22.9	21.1	40
36	57.6	42.1	32.0	26.7	24.3	22.6	41
37	60.3	44.0	33.5	28.2	25.7	24.1	42
38	63.2	46.0	35.1	29.8	27.3	25.6	43
39	66.2	47.9	36.7	21.5	28.9	27.0	44
40	69.6	50.0	38.4	33.2	30.6	28.6	45
41	73.2	52.2	40.2	35.0	32.4	30.3	46
42	76.8	54.5	42.0	36.9	34.3	32.0	47
43	80.4	56.7	43.9	38.7	36.3	33.8	48
44	83.8	58.9	45.7	40.5	38.2	35.7	49
45	86.9	61.0	47.4	42.3	40.2	37.8	50
46	89.9	62.9	49.0	44.2	42.2	40.1	51
47	92.7	64.8	50.8	46.2	44.4	42.5	52
48	95.2	66.5	52.4	48.1	46.7	45.1	53
49	97.2	67.9	54.0	50.1	49.0	47.7	54
50	98.5	69.0	55.5	52.0	51.3	50.3	55
51	99.7	70.1	57.1	54.0	53.5	52.9	56
52	100.9	71.4	58.7	55.9	55.9	55.4	57
53	102.1	73.0	60.4	57.9	58.4	57.9	58
54	103.0	74.4	62.1	59.9	60.8	60.5	59
55	103.3	75.4	63.6	62.0	63.2	63.2	60
56	103.0	75.9	64.9	64.2	65.5	66.2	61
57	102.1	76.4	66.3	66.4	67.7	68.7	62
58	100.8	76.9	67.7	68.3	69.7	70.5	63
59	99.5	77.6	69.2	69.9	71.3	71.7	64
60	98.7	78.4	70.6	71.0	72.3	--	65
61	99.0	79.5	71.9	71.6	--	--	66
62	100.9	81.5	73.1	--	--	--	67
63	104.5	84.4	--	--	--	--	68
64	109.3	--	--	--	--	--	69

Explanatory Notes:

[x] denotes calendar age at entitlement.

$q^{(d)}_{[x]}$ denotes the annual rate of death during the first year of entitlement for those lives who became entitled to disability benefits at age x.

$q^{(d)}_{[x]+i}$ denotes the annual rate of death during the (i+1) year of entitlement for those lives who became entitled to disability benefits at age x.

$q^{(d)}_{x+5}$ denotes the annual rate of death during the year of entitlement commencing at attained age x+5 for those lives who became entitled to disability benefits at age x or younger.

TABLE 9

Graduated Select and Ultimate Death Termination Rates For Female Disabled Workers, 1968-74 OASDI Experience (per thousand)

x	$q^{(d)}_{[x]}$	$q^{(d)}_{[x]+1}$	$q^{(d)}_{[x]+2}$	$q^{(d)}_{[x]+3}$	$q^{(d)}_{[x]+4}$	$q^{(d)}_{x+5}$	x+5
20	27.9	22.5	18.0	14.2	11.0	8.0	25
21	27.7	22.7	18.3	14.5	11.5	8.7	26
22	27.5	22.8	18.5	14.8	12.0	9.4	27
23	27.4	23.0	18.8	15.2	12.5	10.1	28
24	27.4	23.1	19.0	15.5	13.0	10.8	29
25	27.4	23.3	19.2	15.9	13.5	11.5	30
26	27.6	23.6	19.5	16.2	14.0	12.2	31
27	28.0	23.9	19.8	16.7	14.5	12.8	32
28	28.7	24.5	20.3	17.1	15.0	13.4	33
29	29.6	25.2	20.8	17.6	15.4	13.9	34
30	30.8	26.0	21.4	18.1	15.9	14.5	35
31	32.2	27.0	22.1	18.6	16.4	15.0	36
32	33.7	28.0	22.8	19.1	16.8	15.4	37
33	35.4	29.1	23.5	19.6	17.2	15.9	38
34	37.1	30.1	24.1	20.0	17.6	16.2	39
35	39.0	31.2	24.7	20.5	17.9	16.5	40
36	40.9	32.4	25.3	20.9	18.3	16.8	41
37	42.8	33.6	26.1	21.4	18.7	17.1	42
38	44.8	34.9	26.9	22.0	19.1	17.4	43
39	46.9	36.5	27.9	22.7	19.7	17.8	44
40	49.1	38.2	28.9	23.4	20.3	18.3	45
41	51.5	39.9	30.0	24.3	21.0	18.9	46
42	53.8	41.5	31.2	25.1	21.8	19.7	47
43	56.0	43.0	32.2	26.0	22.6	20.6	48
44	58.0	44.2	33.1	26.7	23.5	21.6	49
45	59.6	45.1	33.8	27.3	24.4	22.7	50
46	60.9	45.9	34.4	27.8	25.2	24.0	51
47	62.0	46.5	34.9	28.3	26.1	25.4	52
48	62.9	47.0	35.2	28.7	26.8	26.6	53
49	63.3	47.3	35.5	29.3	27.5	27.6	54
50	63.4	47.4	35.7	29.9	28.3	28.4	55
51	63.4	47.5	36.1	30.7	29.1	29.0	56
52	63.3	47.6	36.5	31.6	30.1	29.8	57
53	63.2	47.7	37.0	32.6	31.1	30.8	58
54	62.9	47.6	37.4	33.4	32.3	32.0	59
55	62.3	47.5	37.7	34.2	33.4	33.3	60
56	61.3	47.2	37.9	35.0	34.5	34.8	61
57	60.1	46.8	38.2	35.7	35.4	36.2	62
58	58.8	46.6	38.6	36.3	36.1	37.2	63
59	57.8	46.5	39.1	36.6	36.4	37.4	64
60	57.6	46.8	39.7	36.9	36.4	--	65
61	58.6	47.8	40.5	37.2	--	--	66
62	60.9	49.5	41.6	--	--	--	67
63	64.0	51.5	--	--	--	--	68
64	67.8	--	--	--	--	--	69

See explanatory notes at bottom of Table 8.

TABLE 10

Graduated Select and Ultimate Recovery Termination Rates
For Male Disabled Workers, 1968-74 OASDI Experience
(per thousand)

x	$q^{(r)}_{[x]}$	$q^{(r)}_{[x]+1}$	$q^{(r)}_{[x]+2}$	$q^{(r)}_{[x]+3}$	$q^{(r)}_{[x]+4}$	$q^{(r)}_{x+5}$	x+5
20	98.8	186.6	108.8	53.6	29.1	18.8	25
21	95.4	179.4	104.1	52.4	29.1	18.8	26
22	91.8	171.9	99.4	51.3	29.1	18.8	27
23	88.0	164.3	94.9	49.9	28.9	18.7	28
24	84.5	157.4	90.7	48.5	28.5	18.5	29
25	81.6	151.2	87.0	46.9	28.0	18.1	30
26	79.2	146.0	83.6	45.1	27.4	17.5	31
27	77.3	141.7	80.6	43.3	26.7	16.8	32
28	75.6	138.1	77.9	41.6	26.0	15.9	33
29	74.2	135.1	75.6	40.1	25.2	14.8	34
30	72.8	132.6	73.8	38.9	24.4	13.8	35
31	71.3	130.1	72.2	37.8	23.7	12.8	36
32	69.7	127.5	70.7	36.9	23.0	12.0	37
33	68.0	124.9	69.1	36.0	22.4	11.3	38
34	66.1	121.9	67.3	35.0	21.7	10.6	39
35	63.8	118.4	65.1	33.7	20.9	9.9	40
36	61.2	114.4	62.7	32.3	20.1	9.3	41
37	58.1	109.9	59.9	30.6	19.1	8.6	42
38	54.8	104.9	56.7	28.8	18.0	8.0	43
39	51.3	99.5	53.2	26.8	16.9	7.3	44
40	47.8	94.0	49.6	24.9	15.6	6.6	45
41	44.7	88.5	45.9	22.9	14.3	5.9	46
42	41.9	83.0	42.3	21.0	13.0	5.3	47
43	39.3	77.3	38.6	19.0	11.6	4.7	48
44	36.7	71.4	35.0	17.1	10.4	4.2	49
45	33.9	65.6	31.5	15.3	9.2	3.7	50
46	31.0	60.0	28.2	13.7	8.1	3.4	51
47	28.1	54.6	25.2	12.2	7.1	3.1	52
48	25.3	49.5	22.4	10.8	6.2	2.8	53
49	22.6	44.9	19.8	9.6	5.4	2.5	54
50	20.1	40.4	17.3	8.5	4.7	2.3	55
51	17.6	36.2	15.0	7.4	4.1	2.1	56
52	15.5	32.2	12.9	6.4	3.6	1.8	57
53	13.5	28.6	11.1	5.5	3.2	1.7	58
54	11.7	25.2	9.6	4.9	2.8	1.5	59
55	10.0	22.0	8.3	4.3	2.5	1.5	60
56	8.5	19.1	7.2	3.8	2.4	1.4	61
57	7.3	16.7	6.5	3.5	2.3	1.4	62
58	6.1	14.5	5.9	3.2	2.3	1.4	63
59	5.0	12.7	5.5	3.1	2.2	1.3	64
60	4.1	11.2	5.1	3.1	2.2	--	65
61	3.2	10.1	5.0	3.0	--	--	66
62	2.5	9.5	5.0	--	--	--	67
63	2.0	9.0	--	--	--	--	68
64	1.7	--	--	--	--	--	69

Explanatory Notes: [x] denotes calendar age at entitlement.

$q^{(r)}_{[x]}$ denotes the annual rate of recovery during the first year of entitlement for those lives who became entitled to disability benefits at age x.

$q^{(r)}_{[x]+i}$ denotes the annual rate of recovery during the (i+1) year of entitlement for those lives who became entitled to disability benefits at age x.

$q^{(r)}_{x+5}$ denotes the annual rate of recovery during the year of entitlement commencing at attained age x+5 for those lives who became entitled to disability benefits at age x or younger.

TABLE 11

Graduated Select and Ultimate Recovery Termination Rates
For Female Disabled Workers, 1968-74 OASDI Experience

x	$q^{(r)}_{[x]}$	$q^{(r)}_{[x]+1}$	$q^{(r)}_{[x]+2}$	$q^{(r)}_{[x]+3}$	$q^{(r)}_{[x]+4}$	$q^{(r)}_{x+5}$	x+5
20	62.7	95.8	60.1	29.3	13.8	5.4	25
21	58.4	91.6	57.3	28.1	13.6	5.9	26
22	54.2	87.4	54.4	26.8	13.4	6.4	27
23	50.3	83.3	51.5	25.6	13.3	6.9	28
24	46.7	79.3	48.8	24.5	13.2	7.4	29
25	43.4	75.5	46.1	23.4	13.0	7.7	30
26	40.2	71.9	43.6	22.4	12.9	8.0	31
27	37.5	68.5	41.5	21.6	12.9	8.1	32
28	35.3	65.6	39.6	20.9	12.7	8.1	33
29	33.6	63.2	38.2	20.5	12.6	8.0	34
30	32.3	61.3	37.1	20.2	12.6	7.8	35
31	31.3	59.7	36.3	20.0	12.6	7.6	36
32	30.6	58.5	35.6	19.8	12.5	7.3	37
33	30.1	57.7	35.0	19.6	12.3	7.0	38
34	29.8	57.1	34.4	19.4	12.1	6.6	39
35	29.4	56.5	33.9	18.9	11.7	6.3	40
36	28.9	56.0	33.4	18.3	11.3	5.9	41
37	28.2	55.4	32.9	17.5	10.7	5.5	42
38	27.3	54.5	32.3	16.6	10.1	5.1	43
39	26.4	53.5	31.6	15.7	9.5	4.7	44
40	25.3	52.3	30.5	14.7	8.9	4.3	45
41	24.1	50.7	29.2	13.8	8.2	4.0	46
42	22.9	48.7	27.6	12.8	7.5	3.8	47
43	21.6	46.3	25.7	11.8	6.8	3.6	48
44	20.5	43.6	23.7	10.9	6.2	3.4	49
45	19.3	40.6	21.5	9.9	5.7	3.1	50
46	19.0	37.5	19.4	9.0	5.2	2.9	51
47	16.4	34.3	17.2	8.1	4.7	2.5	52
48	14.8	31.2	15.2	7.2	4.3	2.2	53
49	13.3	28.2	13.4	6.3	3.8	1.9	54
50	11.9	25.3	11.7	5.6	3.4	1.6	55
51	10.8	22.5	10.2	4.8	3.0	1.4	56
52	9.7	20.0	9.0	4.2	2.6	1.3	57
53	8.6	17.6	7.9	3.7	2.2	1.2	58
54	7.6	15.5	7.0	3.2	2.0	1.2	59
55	6.6	13.7	6.2	2.8	1.8	1.1	60
56	5.8	12.1	5.5	2.5	1.6	1.1	61
57	5.0	10.8	4.9	2.3	1.6	1.1	62
58	4.3	9.7	4.3	2.2	1.5	1.0	63
59	3.6	8.8	3.8	2.1	1.5	.9	64
60	3.0	7.9	3.5	2.1	1.4	--	65
61	2.4	7.1	3.2	2.0	--	--	66
62	1.9	6.7	3.0	--	--	--	67
63	1.5	6.3	--	--	--	--	68
64	1.0	--	--	--	--	--	69

See explanatory notes at bottom of Table 10.

TABLE 12

Ratio of Female to Male Graduated Mortality Rates For Disabled Workers Based on 1968-74 OASDI Experience

Calendar Age at Entitlement	Duration					
	0	1	2	3	4	5 and Over[1/]
20	1.09	1.24	1.43	1.49	1.36	1.07
25	.80	.94	1.06	1.10	1.10	1.04
30	.70	.82	.91	.94	.95	.97
35	.71	.78	.81	.81	.78	.78
40	.71	.76	.75	.70	.66	.64
45	.69	.74	.71	.65	.61	.60
50	.64	.69	.64	.58	.55	.56
55	.60	.63	.59	.55	.53	.53
60	.58	.60	.56	.52	.50	--

1/ Calculated at attained age 5 years older than age at entitlement.

TABLE 13

Ratio of Female to Male Graduated Recovery Rates For Disabled Workers Based on 1968-74 OASDI Experience

Calendar Age at Entitlement	Duration					
	0	1	2	3	4	5 and Over 1/
20	.63	.51	.55	.55	.49	.29
25	.53	.50	.53	.50	.46	.43
30	.44	.46	.50	.52	.52	.57
35	.46	.48	.52	.56	.56	.64
40	.53	.56	.61	.59	.57	.65
45	.57	.62	.68	.65	.62	.84
50	.59	.63	.68	.66	.72	.70
55	.66	.62	.75	.65	.72	.73
60	.73	.71	.69	.68	.64	--

1/ Calculated at attained age 5 years older than age at entitlement.

TABLE 14

Ratio of Disabled Worker Graduated Mortality Rates Based on 1968-74 OASDI Experience to the 1969-71 United States Population Mortality Rates

Attained Calendar Age	Duration 0	1	2	3	4	5 and Over
			Male			
20	11.74	---	---	---	---	---
25	16.22	11.11	7.52	5.44	4.26	3.55
30	20.56	14.16	9.95	7.62	6.12	5.19
35	19.82	13.78	9.89	7.73	6.40	5.36
40	16.65	11.46	8.40	6.75	5.81	5.05
45	13.33	9.03	6.73	5.66	4.97	4.39
50	9.71	6.70	5.17	4.56	4.16	3.73
55	6.44	4.64	3.76	3.48	3.33	3.13
60	4.05	3.19	2.78	2.73	2.69	2.59
			Female			
20	38.48	---	---	---	---	---
25	33.41	28.17	22.93	18.05	14.02	9.76
30	29.06	23.77	19.15	15.75	13.21	10.85
35	24.61	18.99	14.83	12.05	10.35	9.15
40	20.29	15.08	11.12	8.84	7.56	6.82
45	16.15	11.98	8.73	6.80	5.69	4.96
50	11.65	8.69	6.47	5.20	4.63	4.17
55	7.80	5.96	4.63	3.96	3.64	3.56
60	4.98	4.02	3.34	3.09	2.98	2.88

TABLE 15

Select and Ultimate Life Table for Male Disabled Workers
Based on Termination Rates for Death and Recovery,
1968-74 OASDI Experience

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	$l_{[x]+4}$	l_{x+5}	x+5
20	100,000	87,803	70,126	61,708	57,845	55,703	25
21	96,122	84,568	68,047	60,127	56,376	54,243	26
22	92,254	81,354	65,975	58,538	54,900	52,782	27
23	88,476	78,214	63,917	56,934	53,419	51,324	28
24	84,918	75,218	61,893	55,321	51,935	49,875	29
25	81,611	72,383	59,914	53,714	50,462	48,445	30
26	78,569	69,728	57,992	52,125	49,011	47,044	31
27	75,786	67,260	56,142	50,574	47,595	45,678	32
28	73,220	64,963	54,374	49,070	46,219	44,352	33
29	70,867	62,848	52,709	47,631	44,890	43,071	34
30	68,715	60,910	51,159	46,267	43,615	41,838	35
31	66,697	59,097	49,701	44,968	42,389	40,648	36
32	64,776	57,369	48,310	43,722	41,204	39,492	37
33	62,905	55,682	46,949	42,503	40,043	38,357	38
34	61,017	53,985	45,587	41,290	38,892	37,231	39
35	59,067	52,251	44,211	40,075	37,744	36,110	40
36	57,063	50,487	42,828	38,860	36,601	34,996	41
37	55,008	48,688	41,431	37,643	35,461	33,888	42
38	52,921	46,862	40,019	36,426	34,325	32,787	43
39	50,839	45,038	38,614	35,219	33,196	31,695	44
40	48,818	43,250	37,228	34,026	32,078	30,613	45
41	46,888	41,517	35,870	32,849	30,972	29,541	46
42	45,021	39,823	34,531	31,682	29,875	28,476	47
43	43,181	38,150	33,206	30,524	28,786	27,419	48
44	41,350	36,495	31,893	29,372	27,702	26,367	49
45	39,533	34,874	30,599	28,231	26,624	25,319	50
46	37,747	33,290	29,324	27,099	25,548	24,271	51
47	36,003	31,749	28,070	25,973	24,473	23,220	52
48	34,276	30,229	26,822	24,847	23,395	22,165	53
49	32,555	28,727	25,576	23,717	22,312	21,105	54
50	30,832	27,237	24,334	22,586	21,229	20,046	55
51	29,148	25,779	23,104	21,459	20,151	18,993	56
52	27,522	24,363	21,894	20,343	19,083	17,951	57
53	25,967	23,001	20,713	19,245	18,031	16,925	58
54	24,448	21,674	19,557	18,167	16,996	15,918	59
55	22,946	20,370	18,421	17,108	15,979	14,932	60
56	21,474	19,098	17,311	16,071	14,982	13,967	61
57	20,055	17,860	16,221	15,048	14,001	13,023	62
58	18,644	16,664	15,159	14,049	13,047	12,110	63
59	17,314	15,514	14,129	13,081	12,129	11,240	64
60	16,066	14,422	13,143	12,153	11,256	10,420	65
61	14,895	13,377	12,189	11,257	10,420	0	66
62	13,846	12,418	11,298	10,420	0	0	67
63	12,849	11,484	10,420	0	0	0	68
64	11,718	10,420	0	0	0	0	69

Explanatory Note: $[x]$ denotes calendar age at entitlement.

$l_{[x]}$ denotes the number of lives who become entitled to disability benefits at age x.

$l_{[x]+i}$ denotes the number of lives remaining disabled at age x+i of the $l_{[x]}$ lives who became entitled to benefits at age x.

l_{x+5} denotes the number of lives remaining disabled at age x+5 for each life table cohort of lives who became entitled to benefits at age x or younger.

TABLE 16

Select and Ultimate Life Table for Female Disabled Workers Based on Termination Rates for Death and Recovery, 1968-74 OASDI Experience

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	$l_{[x]+4}$	l_{x+5}	x+5
20	100,000	91,113	80,527	74,319	71,115	69,364	25
21	97,401	89,174	79,172	73,275	70,185	68,436	26
22	94,788	87,185	77,752	72,161	69,184	67,436	27
23	92,190	85,156	76,273	70,984	68,115	66,369	28
24	89,621	83,099	74,739	69,743	66,980	65,240	29
25	87,099	81,037	73,170	68,453	65,790	64,057	30
26	84,640	78,990	71,582	67,123	64,552	62,830	31
27	82,304	76,996	70,003	65,771	63,282	61,570	32
28	80,153	75,106	68,462	64,416	61,992	60,291	33
29	78,201	73,336	66,972	63,071	60,695	59,004	34
30	76,406	71,663	65,522	61,736	59,396	57,716	35
31	74,738	70,071	64,109	60,415	58,105	56,436	36
32	73,158	68,533	62,713	59,100	56,821	55,169	37
33	71,644	67,032	61,328	57,794	55,549	53,920	38
34	70,170	65,554	59,952	56,496	54,291	52,694	39
35	68,718	64,100	58,590	55,208	53,056	51,496	40
36	67,299	62,684	57,259	53,943	51,850	50,330	41
37	65,888	61,289	55,953	52,700	50,670	49,193	42
38	64,501	59,927	54,683	51,491	49,522	48,085	43
39	63,153	58,604	53,447	50,318	48,408	47,005	44
40	61,836	57,313	52,243	49,184	47,325	45,952	45
41	60,504	56,009	51,049	48,072	46,261	44,917	46
42	59,137	54,676	49,856	46,972	45,207	43,892	47
43	57,717	53,304	48,654	45,878	44,159	42,867	48
44	56,236	51,888	47,435	44,781	43,110	41,836	49
45	54,671	50,419	46,190	43,669	42,054	40,796	50
46	53,046	48,919	44,925	42,541	40,985	39,744	51
47	51,376	47,398	43,642	41,394	39,900	38,678	52
48	49,688	45,874	42,353	40,238	38,802	37,601	53
49	47,996	44,358	41,069	39,082	37,696	36,519	54
50	46,330	42,873	39,808	37,937	36,598	35,443	55
51	44,739	41,453	38,594	36,822	35,519	34,383	56
52	43,226	40,098	37,426	35,736	34,461	33,339	57
53	41,760	38,785	36,285	34,667	33,415	32,303	58
54	40,311	37,492	35,152	33,602	32,375	31,270	59
55	38,874	36,212	34,021	32,537	31,334	30,235	60
56	37,436	34,938	32,886	31,464	30,287	29,195	61
57	35,999	33,665	31,741	30,378	29,227	28,147	62
58	34,568	32,394	30,585	29,279	28,157	27,100	63
59	33,175	31,144	29,437	28,178	27,090	26,067	64
60	31,874	29,950	28,324	27,107	26,053	25,070	65
61	30,722	28,854	27,279	26,092	25,070	0	66
62	29,642	27,785	26,235	25,070	0	0	67
63	28,458	26,597	25,070	0	0	0	68
64	26,919	25,070	0	0	0	0	69

See explanatory notes at the bottom of Table 15.

TABLE 17

Present Value at Entitlement of a Monthly Annuity Payable To a Disabled Male Worker, 1968-74 OASDI Experience

Calendar Age at Entitlement	Present Value of Annuity at 1/				
	3%	4%	5%	6%	7%
20	126.94	113.22	102.06	92.86	85.19
21	127.42	113.80	102.67	93.48	85.80
22	128.00	114.46	103.37	94.18	86.49
23	128.59	115.13	104.07	94.89	87.19
24	129.00	115.65	104.65	95.49	87.79
25	129.18	115.97	105.05	95.94	88.26
26	129.08	116.04	105.24	96.20	88.55
27	128.68	115.86	105.21	96.26	88.68
28	128.02	115.46	104.97	96.15	88.65
29	127.09	114.82	104.54	95.85	88.46
30	125.90	113.94	103.90	95.38	88.11
31	124.53	112.91	103.11	94.79	87.66
32	123.00	111.74	102.22	94.09	87.12
33	121.30	110.50	101.25	93.33	86.52
34	119.78	109.25	100.28	92.57	85.92
35	118.25	108.70	99.37	91.87	85.38
36	116.79	106.96	98.52	91.22	84.88
37	115.40	105.91	97.72	90.68	84.44
38	114.04	104.88	96.95	90.05	84.02
39	112.66	103.83	96.16	89.46	83.58
40	111.14	102.66	95.26	88.77	83.05
41	109.42	101.30	94.18	87.92	82.39
42	107.53	99.78	92.97	86.94	81.60
43	105.55	98.17	91.66	85.88	80.73
44	103.49	96.50	90.29	84.76	79.82
45	101.36	94.75	88.85	83.57	78.84
46	99.11	92.87	87.29	82.28	77.76
47	96.69	90.84	85.58	80.84	76.55
48	94.15	88.69	83.76	79.29	75.24
49	91.50	86.43	81.83	77.64	73.82
50	88.76	84.07	79.80	75.89	72.32
51	85.81	81.51	77.57	73.96	70.63
52	82.60	78.69	75.09	71.78	68.71
53	79.08	75.56	72.31	69.30	66.51
54	75.30	72.18	69.28	66.58	64.07
55	71.29	68.56	66.01	63.62	61.38
56	66.97	64.62	62.41	60.33	58.38
57	62.28	60.30	58.43	56.66	54.99
58	57.13	55.51	53.98	52.52	51.13
59	51.43	50.16	48.94	47.79	46.68
60	45.07	44.12	43.22	42.35	41.51
61	37.94	37.29	36.67	36.07	35.49
62	29.97	29.58	29.21	28.84	28.49
63	21.12	20.94	20.76	20.59	20.42
64	11.25	11.20	11.16	11.11	11.07

1/ Present value at the time of the first monthly benefit payments calculated on the basis of one unit payment per month during continuance of disability, but not beyond attainment of age 65.

TABLE 18

Present Value at Entitlement of a Monthly Annuity Payable
To a Disabled Female Worker, 1968-74 OASDI Experience

Calendar Age at Entitlement	Present Value of Annuity at 1/				
	3%	4%	5%	6%	7%
20	169.27	148.68	132.18	118.78	107.75
21	169.33	148.99	132.63	119.31	108.32
22	169.38	149.29	133.07	119.83	108.88
23	169.38	149.54	133.48	120.32	109.41
24	169.31	149.73	133.83	120.76	109.91
25	169.14	149.84	134.11	121.15	110.35
26	168.85	149.85	134.30	121.45	110.72
27	168.34	149.66	134.33	121.61	110.96
28	167.47	149.17	134.08	121.54	111.00
29	166.22	148.33	133.54	121.20	110.81
30	164.64	147.22	132.76	120.65	110.42
31	162.78	145.86	131.75	119.90	109.86
32	160.72	144.31	130.59	119.02	109.18
33	158.48	142.62	129.29	118.01	108.40
34	156.12	140.80	127.89	116.92	107.53
35	153.67	138.91	126.42	115.76	106.62
36	151.11	136.92	124.85	114.53	105.63
37	148.47	134.86	123.23	113.24	104.61
38	145.73	132.70	121.52	111.88	103.52
39	142.85	130.41	119.69	110.41	102.33
40	139.83	127.99	117.74	108.83	101.04
41	136.75	125.51	115.73	107.20	99.71
42	133.64	122.99	113.68	105.53	98.34
43	130.51	120.44	111.61	103.83	96.95
44	127.37	117.88	109.51	102.11	95.54
45	124.25	115.33	107.42	100.40	94.13
46	121.10	112.73	105.28	98.64	92.68
47	117.87	110.05	103.07	96.80	91.15
48	114.50	107.24	100.71	94.82	89.51
49	110.96	104.25	98.18	92.69	87.70
50	107.18	101.02	95.43	90.33	85.69
51	103.06	97.46	92.34	87.66	83.38
52	98.57	93.53	88.90	84.65	80.73
53	93.74	89.26	85.12	81.30	77.76
54	88.59	84.66	81.01	77.62	74.47
55	83.11	79.71	76.55	73.59	70.82
56	77.28	74.40	71.70	69.16	66.77
57	71.04	68.66	66.41	64.28	62.28
58	64.35	62.44	60.62	58.90	57.26
59	57.12	55.65	54.24	52.90	51.62
60	49.28	48.21	47.19	46.20	45.25
61	40.80	40.09	39.39	38.73	38.08
62	31.69	31.27	30.86	30.47	30.09
63	21.96	21.76	21.58	21.39	21.21
64	11.47	11.43	11.38	11.33	11.29

1/ Present value at the time of first monthly benefit payment calculated on the basis of one unit payment per month during continuance of disability, but not beyond attainment of age 65.

Appendix: Derivation of a Two-Dimensional Whittaker-Henderson Type B Graduation Formula

By Steven F. McKay, F.S.A. and John C. Wilkin, A.S.A.

The Whittaker-Henderson method of graduation is characterized by an analytic expression measuring the importance of smoothness and fit in a specific numerical graduation. The derivation presented here employs the matrix-vector notation expounded by T.N.E. Greville in his study note on graduation (53.1.73) for the Part 5 examination administered by the Society of Actuaries. In his note Greville gives a derivation of the one-dimensional graduation formula. The derivation of the two-dimensional formula is very similar to that of the one-dimensional formula. Therefore, in order to avoid undue duplication, the following presentation will emphasize the necessary steps to generalize Greville's presentation, without going into any detail on the aspects of this derivation which have been explained sufficiently by Greville.

Our basic problem deals with an array of numbers U'' of m rows and n columns. In this study, the elements of U'' are ungraduated select mortality or recovery rates. Our intuition and experience tells us that these rates should progress smoothly, both within the row (horizontally) and within the column (vertically). In our tabulation of actual data, however, we notice that the smooth progression of rates is upset by fluctuations. Our problem is to obtain an array of graduated rates U, that removes the fluctuations while retaining the underlying pattern of the ungraduated values.

The desired graduated rates must exhibit a certain amount of smoothness and must exhibit the underlying pattern of the ungraduated rates. This means that they must deviate but not too radically from the actual observed rates. In order to measure the amount by which the graduated rates deviate from the ungraduated rates, we sum the squares of the deviations. These deviations are weighted in the graduation procedure in accordance with the exposure observed for each rate, because statistically the deviations from the underlying values vary according to the reciprocal of the exposure. Let W' denote an array of weights, such that the elements of W' correspond to the exposures used in obtaining the observed rates in U''. We, therefore, use the following term to measure fit; the smaller this sum, the closer the fit:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} W'_{ij} \left(U_{ij} - U''_{ij}\right)^2$$

The sum of the squares of second differences are used to measure smoothness. Since we are interested in obtaining a smooth progression of rates both vertically and horizontally, second differences must be calculated both vertically and horizontally. Each column of the graduated U's, will have m-2 second differences. Let Δ_v^2 denote the vertical second difference operator, such that,

$$\Delta_v^2 U_{ij} = U_{(i+2)j} - 2U_{(i+1)j} + U_{ij}$$

Then the term representing a measure of vertical smoothness will be as follows:

$$\sum_{j=1}^{n} \sum_{i=1}^{m-2} (\Delta_v^2 U_{ij})^2 .$$

The smaller the sum, the more the graduated rates will exhibit vertical smoothness.

Each row of the graduated U's will have n-2 second differences. Let Δ_h^2 denote the horizontal second difference operation, such that,

$$\Delta_h^2 U_{ij} = U_{i(j+2)} - 2U_{i(j+1)} + U_{ij}$$

Then the term representing a measure of horizontal smoothness is:

$$\sum_{i=1}^{m} \sum_{j=1}^{n-2} (\Delta_h^2 U_{ij})^2$$

Again, the smaller the sum, the more the graduated rates will exhibit horizontal smoothness.

The above terms for measuring smoothness and fit are then combined and weighted by the constants α and β, giving

$$\sum_{i=1}^{m} \sum_{j=1}^{n} W'_{ij} (U_{ij} - U''_{ij})^2 + \alpha \sum_{j=1}^{n} \sum_{i=1}^{m-2} (\Delta_v^2 U_{ij})^2 + \beta \sum_{i=1}^{m} \sum_{j=1}^{n-2} (\Delta_h^2 U_{ij})^2$$

where α and β indicate the relative emphasis to be placed on vertical and horizontal smoothness, respectively.

It will be noted that the relative emphasis on vertical smoothness need not be the same as the relative emphasis on horizontal smoothness. Indeed, in most graduations of select

tables, the pronounced effect of the selection process will lead to less of a need for smoothness by duration than to smoothness by age. Also, it has been mentioned that our objective is to minimize the combination of the three terms in the expression for smoothness and fit. There is a certain amount of dependency among each of these three terms. Lowering the value of any one of the terms results in raising the value of at least one of the other two. The set of values U which minimize this expression are regarded as the graduated values and result in the best combination of smoothness and closeness of fit for the given constants α and β.

The first step in minimizing the above expression is to write the expression using matrix-vector notation. We begin by creating the column vector u" obtained by arranging the rows of the array of ungraduated values U", vertically into one column vector of length m x n. Thus, $u"_z$ will correspond to $U"_{ij}$ where $z = n(i-1)+j$. Let u denote the corresponding column vector of graduated values. When we have obtained the graduated values of u our problem will be solved, since the values of U are simply the values of u rearranged into an array.

Let us point out the elementary fact, that if x is a column vector, whose elements are $x_1, x_2, \ldots, x_s$, and if x^T denotes its transpose, then the product

$$x^T x = \sum_{i=1}^{s} x_i^2 .$$

More generally, if K is an s x s diagonal matrix whose diagonal elements are $k_1, k_2, \ldots, k_s$, then

$$x^T Kx = \sum_{i=1}^{s} k_i x_i^2$$

Now, let us define the mn x mn diagonal matrix W whose diagonal elements are the corresponding weights of the elements of u". In other words, W is obtained by arranging the rows of the array W' one right after the other along the diagonal. Therefore, the value of W_{zz} will correspond to the value of W'_{ij}, where $z = n(i-1)+j$.

It is now a relatively simple task to write our expression for fit using matrix-vector notation as follows:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} W'_{ij} (U_{ij} - U"_{ij})^2 = (u-u")^T W(u-u")$$

Before obtaining our matrix expressions for smoothness, however, it must be emphasized that the earlier definitions of vertical and horizontal second differences still apply. As an example, if m = 50 and n = 6, then

$$\Delta_v^2 u_{27} = \Delta_v^2 U_{5,3} = U_{7,3} - 2U_{6,3} + U_{5,3} = u_{39} - 2u_{33} + u_{27} \quad \text{and}$$

$$\Delta_h^2 u_{27} = \Delta_h^2 U_{5,3} = U_{5,5} - 2U_{5,4} + U_{5,3} = u_{29} - 2u_{28} + u_{27}$$

Now, let us point out that there is an mn x mn matrix V such that the elements of the vector Vu are the n(m-2) vertical second differences of the elements of u and 2n zeros, and an mn x mn matrix H such that the elements of the vector Hu are the m(n-2) horizontal second differences of the elements of u and 2m zeros.

The expressions for smoothness can thus be written.

$$\alpha(Vu)^T(Vu) + \beta(Hu)^T(Hu) = \alpha u^T V^T Vu + \beta u^T H^T Hu.$$

As an example, for m = n = 5, the matrices V, H, V^TV, and H^TH are shown on the following pages.

H =

Row/Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

V =

Row/Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$V^T V =$

Row/Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
6	-2	0	0	0	0	5	0	0	0	0	-4	0	0	0	0	1	0	0	0	0	0	0	0	0	0
7	0	-2	0	0	0	0	5	0	0	0	0	-4	0	0	0	0	1	0	0	0	0	0	0	0	0
8	0	0	-2	0	0	0	0	5	0	0	0	0	-4	0	0	0	0	1	0	0	0	0	0	0	0
9	0	0	0	-2	0	0	0	0	5	0	0	0	0	-4	0	0	0	0	1	0	0	0	0	0	0
10	0	0	0	0	-2	0	0	0	0	5	0	0	0	0	-4	0	0	0	0	1	0	0	0	0	0
11	1	0	0	0	0	-4	0	0	0	0	6	0	0	0	0	-4	0	0	0	0	1	0	0	0	0
12	0	1	0	0	0	0	-4	0	0	0	0	6	0	0	0	0	-4	0	0	0	0	1	0	0	0
13	0	0	1	0	0	0	0	-4	0	0	0	0	6	0	0	0	0	-4	0	0	0	0	1	0	0
14	0	0	0	1	0	0	0	0	-4	0	0	0	0	6	0	0	0	0	-4	0	0	0	0	1	0
15	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	6	0	0	0	0	-4	0	0	0	0	1
16	0	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	5	0	0	0	0	-2	0	0	0	0
17	0	0	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	5	0	0	0	0	-2	0	0	0
18	0	0	0	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	5	0	0	0	0	-2	0	0
19	0	0	0	0	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	5	0	0	0	0	-2	0
20	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-4	0	0	0	0	5	0	0	0	0	-2
21	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-2	0	0	0	0	1

$H^T H =$

Row/Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	-2	5	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	-4	5	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	-2	5	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	1	-4	5	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	-2	5	-4	1	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	1	-4	5	-2	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	5	-4	1	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	5	-2	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	5	-4	1	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	5	-2
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1

The analytical matrix expression which combines both smoothness and fit is:

$$(u-u")^T W(u-u") + \alpha u^T V^T V u + \beta u^T H^T H u$$

This expression can be minimized in a manner parallel to that used by Greville. Using this approach, we let $A = W + \alpha V^T V + \beta H^T H$. We then use the fact that A is a positive definite matrix to prove that our expression for smoothness and fit assumes its minimum value when $Au = Wu"$. This matrix vector equation represents an mn x mn system of linear equations in the mn elements of u. Solving these equations for u gives us our desired graduated values.

It should be pointed out that a square matrix is called symmetric if it is equal to its transpose. A symmetric matrix K is called positive definite if the scalar quantity $x^T K x$ is positive for every vector x that is not a vector of zeros. It can be shown that if K is positive definite, then it is non-singular, and therefore has an inverse, and its inverse K^{-1} is also positive definite. Note that A is symmetric. It is also positive definite, assuming α, β, and the weights W_{xx} are all positive, because

$$x^T A x = x^T W x + \alpha (Vx)^T Vx + \beta (Hx)^T Hx$$

$$= \sum_{i=1}^{mn} W_i x_i^2 + \alpha \Sigma (\Delta_v x_i)^2 + \beta \Sigma (\Delta_h x_i)^2$$

All summation terms are non-negative, and the first summation is positive unless all the elements x_i are zero.

Since A is positive definite, it is non-singular and A^{-1} is positive definite. Therefore, the original expression for smoothness and fit can be manipulated as follows:

$$(u-u")^T W(u-u") + \alpha u^T V^T V u + \beta u^T H^T H u$$

$$= (u^T - u"^T)(Wu - Wu") + \alpha u^T V^T V u + \beta u^T H^T H u$$

$$= u^T Wu - u"^T Wu - u^T Wu" + u"^T Wu" + \alpha u^T V^T V u + \beta u^T H^T H u$$

$$= u^T (W + \alpha V^T V + \beta H^T H) u - u"^T Wu - u^T Wu" + u"^T Wu"$$

$$= u^T Au - u''^T Wu - u^T Wu'' + u''^T Wu''$$

$$= u^T AA^{-1} Au - u''^T WA^{-1} Au - u^T AA^{-1} Wu'' + u''^T Wu''$$

$$= (u^T A - u''^T W)A^{-1} (Au - Wu'') + u''^T Wu'' - u''^T WA^{-1} Wu''$$

$$= (Au - Wu'')^T A^{-1} (Au - Wu'') + u''^T Wu'' - u''^T WA^{-1} Wu''$$

Since only the first term involves the unknown vector u, the second and third terms are constant. Therefore, the analytical expression for fit and smoothness is minimized when the first term is smallest. But, since A^{-1} is positive definite, the first term is non-negative, and its smallest value is zero. This value occurs when Au = Wu".

The best method for solving this system of equations is the square-root or Choleski method. In this method the matrix A is factored into the product of a lower-triangular and an upper-triangular matrix, each the transpose of the other. This is possible because A is positive definite. A triangular system of linear equations is, of course, very easy to solve by successive substitution.

First, the linear system Au=Wu" is written as $LL^Tu=Wu''$. The elements of L must be computed such that $A=LL^T$. This is done one row at a time. If $A=(a_{ij})$ and $L=(l_{ij})$, the formulas are as follows:

$$l_{11} = \sqrt{a_{11}}$$

$$l_{ij} = \frac{1}{l_{jj}}\left(a_{ij} - \sum_{h=1}^{j-1} l_{ih}\, l_{jh}\right) \quad (i=2,3,\ldots,nm;\ j=1,2,\ldots,i-1)$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{h-1}^{i-1} l^2_{ih}} \quad (i=2,3,\ldots,mn)$$

Then, let $u'=L^Tu$. Solve the system Lu'=Wu" for the vector u', and then the system $L^Tu=u'$, for the desired vector u.

Since the size of the A and L matrices can easily fill the core of many computers, it is often essential to utilize the band property of these matrices when programming this method of graduation. This can be done by rotating the diagonal lines into vertical lines, and then dropping the columns outside of the band.

A few important properties of the graduated rates that are obtained by the two-dimensional Type B formula are worth mentioning. These properties can be represented by the following three equations:

$$(1)\quad \sum_{i=1}^{m}\sum_{j=1}^{n} U_{ij} W'_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} U''_{ij} W'_{ij}$$

$$(2)\quad \sum_{i=1}^{m}\sum_{j=1}^{n} iU_{ij} W'_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} iU''_{ij} W'_{ij}$$

$$(3)\quad \sum_{i=1}^{m}\sum_{j=1}^{n} jU_{ij} W'_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} jU''_{ij} W'_{ij}$$

If the weights are equal to the exposures to termination, the first equation assures that the total number of graduated and ungraduated terminations will be equal. If it is further assumed that n equals the maximum duration (i.e., there is no ultimate column), the last two equations assure that the average age at entitlement and the average duration at terminaton for both the graduated and ungraduated terminations will be equal.